

Generalized MacMahon $G_d(q)$ as q -deformed CFT_2 Correlation Function

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March 3, 2008

Abstract

Using $\Gamma_{\pm}(z)$ vertex operators of the $c = 1$ two dimensional conformal field theory, we give a 2d-quantum field theoretical derivation of the conjectured d -dimensional MacMahon function $G_d(q)$. We interpret this function $G_d(q)$ as a $(d+1)$ - point correlation function $\mathcal{G}_{d+1}(z_0, \dots, z_d)$ of some local vertex operators $\mathcal{O}_j(z_j)$. We determine these operators and show that they are particular composites of q -deformed hierarchical vertex operators $\Gamma_{\pm}^{(p)}$, with a positive integer p . In agreement with literature's results, we find that $G_d(q)$, $d \geq 4$, cannot be the generating functional of all d - dimensional generalized Young diagrams .

Key words: *Topological string, vertex operators, Young diagrams and solid partitions, $c = 1$ 2d conformal field model, Generalized MacMahon function, q -deformed QFT_2 .*

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1 Introduction

The study of two dimensional (2d) MacMahon function G_2 , and its *3d- generalization* G_3 appear in many areas of statistical physics, such as crystals growth, crystals melting, Bose-Einstein statistics and dimer model [1]-[7]. Recently, these functions have known a revival of interest in connection with topological string theory [8, 9, 10]; in particular in the study of BPS black holes, given by branes wrapping collapsed cycles in Calabi-Yau orbifolds, and in the infinite n limit of quiver gauge theories [11, 12, 13, 14, 15]. MacMahon functions $G_2(q)$ and $G_3(q)$ are also used in the explicit computation of the amplitudes of A-model topological string on local Calabi-Yau manifolds [16, 17, 5, 18].

In [5], it has been shown that the topological A-model partition function Z_{3d} on the complex space \mathbb{C}^3 , which coincides exactly with the 3d- crystal melting partition function $Z_{crystal}$, is given by *3d- generalized* MacMahon function G_3 . This is an important result since topological amplitudes for the full class of toric Calabi-Yau threefolds X_3 with *a planar toric geometry* are recovered just by gluing the \mathbb{C}^3 -vertices [16]. Amplitudes involving open strings are also recovered up on inserting special Lagrangian D- branes

captured by boundary conditions on the edges of the 3- vertices [18]. An evidence for a "topological 4- vertex", in the case of toric Calabi-Yau threefold with *non planar toric geometry*, has been also studied in [19] and would, roughly, be described by a *4d-extension* of the generalized MacMahon function G_3 .

MacMahon functions G_2 and G_3 appear as well in representation theory of infinite dimensional Lie algebras and in topologically twisted $U(1)$ gauge theories [20, 21, 22, 23]. They are respectively the (specialized) character ch_R of the basic representations of the $sl(\infty)$ and the affine algebra $\widehat{sl(\infty)}$ at large central charge c [24, 25, 22, 26]. In the limit $c \rightarrow \infty$, it has been moreover observed in [27] that the basic representation of $\widehat{sl(\infty)}$ is closely related to the partition function of a three dimensional free field theory. MacMahon G_2 and G_3 are also the partition functions of the 4d- and 6d- topologically twisted $U(1)$ gauge theory given by a D4 and D6-branes filling respectively \mathbb{C}^2 and \mathbb{C}^3 [15, 11].

For higher dimensions, it has been checked in [28] that $G_4(q)$ cannot be the generating functional of the *4d-generalized* Young diagrams leading then to the two basic questions:

- (i) what is the right generating functional of generalized *d-dimensional* Young diagrams for $d \geq 4$.
- (ii) what is the exact interpretation of $G_4(q)$ and $G_d(q)$ in general.

These questions are not trivial and their answer needs developing more involved mathematical machinery. Nevertheless, a first step towards the answer of these questions is to start by deepening the study of the conjectured generalized *d-dimensional* MacMahon function G_d . In particular the issues regarding its interpretation in 2d- quantum field theory and its explicit derivation using correlation functions of local vertex operators.

A natural way to reach this goal is to use the "transfer matrix" approach [5, 18, 29] and borrow ideas from q-deformed QFT₂ [30, 31, 32]. This method has been successfully used for the particular case $d = 3$ and could, à priori, extended to higher d- dimensions. In topological string on local Calabi-Yau threefolds, the key idea in getting topological closed string partition function relies on expressing Z_{3d}^{closed} as a particular vev $\langle 0|\mathcal{T}|0\rangle$ of some hermitian transfer matrix operator \mathcal{T} . This operator can be factorized as $\mathcal{A}_+\mathcal{A}_-$ with \mathcal{A}_+ and \mathcal{A}_- being *composite* local vertex operators of the two dimensional $c = 1$ conformal field theory. Implementation of open strings leads to $Z_{3d}^{open} \sim C_{\nu\mu\lambda}$ and is achieved as $\langle \nu^t | \mathcal{A}_+(\lambda) \mathcal{A}_-(\lambda^t) | \mu \rangle$ by inserting boundary states $|\sigma\rangle$, $\sigma = \nu, \lambda, \mu$, described by asymptotic 2d- Young diagrams. If we let string interpretation aside, this construction could be applied as well for higher d- dimensions where G_d is expected to play a central role.

This paper has two main objectives:

- (1) Give a conformal field theoretical derivation of the conjectured *d-dimensional* generalized MacMahon function G_d expressed by the following formula,

$$G_d(q) = \prod_{k=1}^{\infty} \left[(1 - q^k)^{-\frac{(k+d-3)!}{(k-1)!(d-2)!}} \right], \quad d \geq 2,$$

together with the two special ones filling the hierarchy,

$$G_1(q) = \frac{1}{1-q} \quad , \quad G_0(q) = 1. \quad (1.1)$$

Recall that in combinatorial analysis, the function $G_3(q)$ can be defined as the generating functional of *3-dimensional* partitions $\Pi^{(3)}$ extending the usual 2d- partitions $\mu = \Pi^{(2)}$ to higher 3-dimensions. Refined studies regarding G_4 function have revealed that it is not the generating functional of 4d partitions [28, 33, 18].

To fix the ideas, it is interesting to recall that expanding $G_2(q)$ as a q^n power series like,

$$G_2(q) = \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k} \right) = \sum_{n=0}^{\infty} p_2(n) q^n,$$

one gets the number $p_2(n)$ of 2d- partitions (Young diagrams) containing n boxes. From this view, G_2 can be physically interpreted as the exact partition function $Z_2 = \text{Tr}(q^H)$ of a two dimensional statistical physics system with,

$$q = \exp \left(-\frac{1}{KT} \right),$$

and energy spectrum $E_k = k$. Here T is the absolute temperature and the constant K is the Boltzmann one. For instance, $G_2(q)$ is the partition function of the $c = 1$ free Bose gaz. There, the Hamiltonian is given by $\mathcal{H} = \hbar\omega \sum k \mathcal{N}_k$ with $\mathcal{N}_k = a_k^+ a_k$ being the operator number of particles and energy spectrum $E_k = \hbar\omega k$.

Similar expansions can be also made for $G_d(q)$ which then read as follows

$$G_d(q) = \sum_{n=0}^{\infty} p_d(n) q^n, \quad d \geq 3.$$

For the case $d = 3$, the number $p_3(n)$ is precisely the number of 3d- partitions; but for $d = 4$, the number $p_4(n)$ is not the total number of 4d- partitions as it has been explicitly checked in [28].

(2) The second objective of the present study is to show that $G_d(q)$ can be remarkably interpreted as a $(d+1)$ - point correlation function \mathcal{G}_{d+1} of some q - *deformed* vertex operators $\mathcal{O}_j(x_j)$, i.e

$$G_d(q) = \mathcal{G}_{d+1}(x_0, x_1, x_2, \dots, x_d) \quad (1.2)$$

with $x_j = q^j$, $j = 0, \dots, d$; and

$$\mathcal{G}_{d+1} = \langle 0 | \mathcal{O}_0(x_0) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \dots \mathcal{O}_d(x_d) | 0 \rangle. \quad (1.3)$$

The $\mathcal{O}_j(x_j)$'s will be determined in terms of the usual vertex operators $\Gamma_{\pm} = \exp(\Phi_{\pm})$ of the $c = 1$ two dimensional bosonic conformal field theory [34]; but also others, denoted like $\Gamma_{\pm}^{(p)}$, involving q -deformed QFT₂. This result gives:

(i) a q -*deformed* 2d quantum field theoretical proof of the conjectured MacMahon function G_d ,

- (ii) an interpretation of G_d using q -deformed $c = 1$ conformal field theory rather than CFT_2 free field theory with central charge $c \rightarrow \infty$.
- (iii) For $d \geq 4$ G_d cannot be the generating function of d -generalized partitions; but rather of a subclass of d -partitions with very specific boundary conditions.

The organization of this paper is as follows:

In section 2, we introduce the usual vertex operators Γ_{\pm} of the $c = 1$ 2d conformal model and give some of their properties essential for the next steps. In section 3, we revisit the CFT_2 derivation of 3d-generalized MacMahon function G_3 using transfer matrix method. We also introduce the q -deformed $\Gamma_{\pm}^{(2)}$ vertex operators. In section 4, we derive the generalized MacMahon function G_n for 4d and 5d using transfer matrix method and q -deformed vertex operators $\Gamma_{\pm}^{(3)}$ and $\Gamma_{\pm}^{(4)}$. In section 5, we give the result for generic d -dimensions. In section 6, we derive $\mathcal{O}_j(x_j)$ vertex operators involved in $G_d(q)$ re-interpreted as $(d+1)$ -point correlation function $\mathcal{G}_{d+1}(z_0, \dots, z_d)$ in q -deformed $c = 1$ CFT_2 . In the conclusion section, we summarize the main results of the paper accompanied with a discussion. In appendices A and B, we give more details on the proofs of identities used in the present study.

2 Vertex operators: useful properties

In this section, we explore some basic properties of the vertex operators $\Gamma_{\pm}(z)$ in $c = 1$ 2d-conformal field theory. We study their commutation relations algebra in connection with the counting of the Hilbert space states and the 2d-partitions (Young diagrams). We also give special features of $\Gamma_{\pm}(z)$ which has motivated us to look for the relations (1.2-1.3).

2.1 Vertex operators in $c = 1$ CFT_2

As the $c = 1$ field vertex operators $\Gamma_{\pm}(z)$, $z \in C$, have been well studied and are quite known in 2d conformal field theory [35, 36, 29], we shall come directly to the main points by considering the three following materials needed for the study of $\Gamma_{\pm}(z)$ and their extensions to be considered in this study:

(1) $U(1)$ Kac-Moody algebra.

In CFT_2 on the complex line C parameterized by the coordinate z , the $U(1)$ Kac-Moody algebra is generated by the holomorphic current $J(z)$ obeying the following operator product expansion (OPE)

$$J(z_1) J(z_2) = \frac{1}{(z_1 - z_2)^2} + \text{regular terms.} \quad (2.1)$$

Using the Laurent expansion,

$$J(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J_n, \quad J_n = \oint \frac{dz}{2i\pi} z^n J(z), \quad (2.2)$$

the above OPE algebra reads as follows

$$[J_n, J_m] = n\delta_{n+m,0}. \quad (2.3)$$

We have, amongst others, $(J_n)^\dagger = J_{-n}$ and $J_n |0\rangle = 0$ for $n \geq 1$.

(2) $c=1$ conformal model.

In the 2D conformal field theoretic realization of eqs(2.1-2.3), one distinguishes two free field theoretic realizations of the $c = 1$ conformal representation:

(i) The free bosonic realization using a single real (chiral) boson $\Phi(z)$ with the usual two- point correlation function,

$$\Phi(z_1)\Phi(z_2) = -\ln(z_1 - z_2) + \text{regular terms}. \quad (2.4)$$

(ii) The free fermionic realization using a complex one component fermion $\psi(z)$. In this case, the two- point correlation function that have a singular term is,

$$\psi^*(z_1)\psi(z_2) = \frac{1}{z_1 - z_2} + \text{regular terms}. \quad (2.5)$$

The two- point functions $\psi(z_1)\psi(z_2)$ and $\psi^*(z_1)\psi^*(z_2)$ are regular.

The U(1) Kac-Moody current $J(z)$ is given, in the bosonic representation, by:

$$J(z) = \frac{\partial\Phi(z)}{i\partial z}, \quad (2.6)$$

while it has the following form $J(z) = i\psi^*(z)\psi(z)$ in terms of fermions. Below, we shall mainly focus on the bosonic case; the link with fermionic representation can be done by using bosonization ideas.

Expanding the 2d chiral scalar field as

$$\Phi(z) = \sum_{n \in \mathbb{Z}} z^{-n} \Phi_n \quad (2.7)$$

and rearranging it as $\Phi(z) = \Phi_-(z) + \Phi_0 + \Phi_+(z)$, we can write the above expansion as,

$$\Phi_-(z) = i \sum_{n>0} \frac{1}{n} z^n J_{-n}, \quad \Phi_+(z) = -i \sum_{n>0} \frac{1}{n} z^{-n} J_n \quad (2.8)$$

where we have used

$$\Phi_n = \frac{1}{in} J_n, \quad n \in \mathbb{Z}^*. \quad (2.9)$$

This identity follows directly by comparing eq(2.6-2.7) and (2.2). Notice also that the zero mode Φ_0 acts trivially; it will be ignored in follows.

(3) Vertex operators: Level 1.

There are various local field vertex operators that we will encounter in this present study. The simplest ones, named as *level 1*, are given by

$$\Gamma_\pm(z) = \exp \Phi_\pm(z), \quad z \in C, \quad (2.10)$$

The other vertex operators $\Gamma_{\pm}^{(p)}(z)$, to be introduced later on, will be named as *level p vertex operators*. Substituting $\Phi_{\pm}(z)$ by their expression (2.8), the *Level 1* vertex operators ($\Gamma_{\pm}(z) \equiv \Gamma_{\pm}^{(1)}(z)$) read also as follows

$$\begin{aligned}\Gamma_{-}(z) &= \exp\left(i \sum_{n>0} \frac{1}{n} z^n J_{-n}\right) , \\ \Gamma_{+}(z) &= \exp\left(-i \sum_{n>0} \frac{1}{n} z^{-n} J_n\right) .\end{aligned}\tag{2.11}$$

These objects may be interpreted as the generating functionals of monomials of the J_m operators. For instance, we have for the leading terms,

$$J_{-1} = \frac{\partial \Gamma_{-}(0)}{i \partial z} , \quad (J_{-1}^2 - i J_{-2}) = \frac{\partial^2 \Gamma_{-}(0)}{(i \partial z)^2}\tag{2.12}$$

and similar relations for their adjoints. Notice that since the states of the Hilbert space of $c = 1$ conformal theory representation are given by

$$\prod_{i \geq 1} (J_{-n_i})^{\lambda_i} |0\rangle , \quad J_{n_i} |0\rangle = 0, \quad n_i, \lambda_i \in N ,\tag{2.13}$$

it follows that the state

$$\Gamma_{-}(z) |0\rangle ,\tag{2.14}$$

is the generating functional of the basis states (2.13) of the $c = 1$ CFT₂ Hilbert space.

2.2 Algebra of the Γ_{\pm} vertex operators

The action of the local operators $\Gamma_{\pm}(z)$ on the Hilbert space states of the $c = 1$ 2d-conformal field theory exhibits a set of special properties inherited from the algebra of the $J_{\pm n}$ modes (2.3). Some of these properties are revisited in what follows:

(1) The $\Gamma_{\pm}(z)$ operators obey the algebra,

$$\begin{aligned}\Gamma_{\pm}(x) \Gamma_{\pm}(y) &= \Gamma_{\pm}(y) \Gamma_{\pm}(x), \quad x, y \in C, \\ \Gamma_{+}(x) \Gamma_{-}(y) &= \left(1 - \frac{y}{x}\right)^{-1} \Gamma_{-}(y) \Gamma_{+}(x),\end{aligned}\tag{2.15}$$

which can be easily established by using eqs(2.4-2.3).

(2) The operator q^{L_0} acts on $\Gamma_{+}(1)$ and $\Gamma_{-}(1)$ as a translation operator as shown below,

$$q^{L_0} \Gamma_{\pm}(1) q^{-L_0} = \Gamma_{\pm}(q).\tag{2.16}$$

This relation will play a crucial role later on, in particular when using transfer matrix method.

(3) Using the properties $J_n |0\rangle = 0$ and $\langle 0 | J_{-n} = 0$ for $n > 0$, we have moreover:

(i) For all positions z , the operators $\Gamma_{\pm}(z)$ act on the vacuum as the identity operator:

$$\Gamma_{+}(z) |0\rangle = |0\rangle, \quad \langle 0 | \Gamma_{-}(z) = \langle 0 |.\tag{2.17}$$

So we have

$$\langle 0|\Gamma_-(z)|0 \rangle = 1, \quad \langle 0|\Gamma_+(z)|0 \rangle = 1, \quad (2.18)$$

and

$$\begin{aligned} \langle 0|\Gamma_-(z)\Gamma_+(w)|0 \rangle &= 1, \\ \langle 0|\Gamma_\pm(z)\Gamma_\pm(w)|0 \rangle &= 1. \end{aligned} \quad (2.19)$$

Notice in passing that, viewing $\Gamma_\pm(z)$ as local operator fields, eqs(2.18) and (2.19) may be interpreted respectively as 1- point and 2- point Green functions. Notice moreover that since $\Gamma_+(z)$ and $\Gamma_-(w)$ are non commuting operators, we have,

$$\langle 0|\Gamma_+(z)\Gamma_-(w)|0 \rangle \neq \langle 0|\Gamma_-(z)\Gamma_+(w)|0 \rangle. \quad (2.20)$$

We will develop this issue much more later when we come to the derivation of eqs(1.2-1.3) by using correlation functions.

(ii) As $\Gamma_-(z)$ involves all monomials in J_{-n_i} ,

$$J_{-\mathbf{n}}^\lambda \equiv \prod_{i \geq 1} (J_{-n_i})^{\lambda_i}, \quad (2.21)$$

where $\lambda = (\lambda_1, \lambda_2, \dots)$ is a 2d- partition, the state $\Gamma_-(z)|0 \rangle$ is reducible and is given by a sum over all possible 2d- partitions λ . In particular we have for $z = 1$,

$$\Gamma_-(1)|0 \rangle = \sum_{\text{2d partitions } \lambda} |\lambda \rangle. \quad (2.22)$$

A similar relation is also valid for $\langle 0|\Gamma_+(1)$. More generally, this relation extends as $\Gamma_-(1)|\mu \rangle$ and involves the Schur function $S_\mu^{\text{Schur}}(q)$ [35]. With these tools, we are in position to proceed for higher dimensional generalizations.

3 The 3d- MacMahon function revisited

Our main objectives here are:

- (i) revisit the derivation of Z_{3d}
- (ii) use CFT₂ explicit computations to give arguments which support the existence of a hierarchy of *level p* vertex operators $\Gamma_\pm^{(p)}$.

To reach this goal, we first give some details on 3d- partitions (known also as plane partitions) and its generating functional Z_{3d} . Then we present the explicit computation of the function Z_{3d} using transfer matrix method. As mentioned in the introduction, Z_{3d} is precisely the amplitude of the topological 3- vertex of closed strings on \mathbb{C}^3 . There, the q - parameter is given by

$$q = \exp(-g_s), \quad (3.1)$$

with g_s being the topological string coupling constant. Z_{3d} is also the partition function of corner melting 3d- crystals.

3.1 Plane partitions and 3d- Hilbert states

To begin notice that, from the view of combinatory analysis, the 3d- MacMahon function G_{3d} can be defined by the following partition function

$$Z_{3d} = \sum_{3d \text{ partitions } \Pi^{(3)}} q^{|\Pi^{(3)}|} , \quad (3.2)$$

where $|\Pi^{(3)}|$ is the number of boxes of the 3d- generalized Young diagram. This relation may be also written as

$$Z_{3d} = \sum_{3d \text{ partitions } \Pi^{(3)}} \langle \Pi^{(3)} | q^{\mathcal{H}} | \Pi^{(3)} \rangle , \quad (3.3)$$

with

$$\mathcal{H} |\Pi^{(3)}\rangle = E |\Pi^{(3)}\rangle , \quad E = |\Pi^{(3)}| . \quad (3.4)$$

The Hilbert space states $|\Pi^{(3)}\rangle$, to which we shall refer as "3d- Hilbert states", are the quantum states associated with $\Pi^{(3)}$. The relation (3.2) has a remarkable combinatorial interpretation; it is the generating function of the $p_3(n)$ number of 3d- partitions $\Pi^{(3)}$ with n boxes. The $p_3(n)$ number can be determined by expanding Z_{3d} like,

$$Z_{3d}(q) = \sum_{n=0}^{\infty} p_3(n) q^n , \quad p_3(n) = \frac{\partial^n Z_{3d}(0)}{n! \partial q^n} . \quad (3.5)$$

Notice also that 3d- partitions $\Pi^{(3)}$ are 3d- generalizations of Young diagrams and can be decomposed as a sequence¹ of 2d- partitions $\Pi_t^{(2)}$ like,

$$\Pi^{(3)} = \sum_{t \in Z} \Pi_t^{(2)} , \quad (3.6)$$

where t parameterizes the slices. For fixed integer t , the 2d-partition $\Pi_t^{(2)} = (\Pi_{a,a+t})_{a \in \mathbb{N}^*}$ lives on the diagonal plane $b = a + t$ of the cubic lattice $\mathbb{N}^* \times \mathbb{N}^* \times \mathbb{N}^*$ parameterized by the positive integers (a, b, c) . The diagonal decomposition (3.6) is useful here in the sense it is used in the transfer matrix method for computing Z_{3d} . There exist an other decomposition of $\Pi^{(3)}$ namely the so called perpendicular decomposition relevant for the study of the topological vertex.

Expressing the number $|\Pi^{(3)}|$ of boxes of 3d -partition in terms of 2d ones, namely

$$|\Pi^{(3)}| = \sum_t |\Pi_t^{(2)}| , \quad (3.7)$$

¹3d- partitions $\Pi^{(3)}$ have integer entries ($\Pi_{a,b} \geq 0$) such that $\Pi_{a,b} \geq \Pi_{a+i,b+j}$ $i, j \geq 0$. These are 3d generalizations of the usual Young diagrams described by the 2d- partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_a \geq \lambda_{a+1}$. The partitions $\Pi^{(3)}$ have several properties; in particular the diagonal slicing in terms of 2d partitions $\Pi_t^{(2)}$ used in the transfer matrix method. The diagonal slicing of $\Pi^{(3)} = (\Pi_{a,b})$ is obtained by setting $b = a + t$ where $t \in Z$ parameterizes the sequence $\Pi_t^{(2)}$. For fixed t , $\Pi_t^{(2)}$ may be thought of as λ with parts $\lambda_a = \Pi_{a,a+t}$.

we can put eq(3.2) in the form

$$Z_{3d} = \sum_{3d \text{ partitions } \Pi^{(3)}} \left(\prod_t q^{|\Pi_t^{(2)}|} \right). \quad (3.8)$$

To get "3d- generalized Hilbert states" $|\Pi^{(3)}\rangle$, it is interesting to first recall 2d- generalized Hilbert space states $|\Pi^{(2)}\rangle \equiv |\lambda\rangle$. In the language of the U(1) Kac-Moody algebra representations, the Hilbert space states of the $c = 1$ CFT₂ have the structure

$$|\lambda\rangle = |\lambda_1, \dots, \lambda_i, \dots\rangle, \quad (3.9)$$

and are completely characterized by 2d- partitions,

$$\lambda = (\lambda_1, \dots, \lambda_i, \dots), \quad \lambda_1 \geq \lambda_2 \geq \dots, \quad \lambda_i \in \mathbb{N}. \quad (3.10)$$

The generating functional of these states is given by $\Gamma_-(1)|0\rangle = \sum_{\lambda} |\lambda\rangle$, eq(2.22). Generalized Hilbert space states $|\Pi^{(3)}\rangle$, associated to 3d- partitions $\Pi^{(3)}$ may be built out 2d - partitions with interlacing relations [5].

The generating functional of 3d partitions requires, in the framework of transfer method, the following ²

$$\Psi_-(1)|0\rangle = \left(\prod_{t=-\infty}^{-1} (\Gamma_-(1) q^{L_0}) \right) |0\rangle, \quad (3.11)$$

together with

$$\langle 0|\Psi_+(1) = \langle 0| \left(\prod_{t=0}^{\infty} (q^{L_0} \Gamma_+(1)) \right). \quad (3.12)$$

Notice in passing that in eqs(3.11-3.12), the products $\prod_{t=-\infty}^{-1} (\dots)$ and $\prod_{t=0}^{\infty} (\dots)$ are taken over diagonal slices of the 3d partitions. These products are typical ones in the transfer matrix method where a 3d partition is thought of as a bound state from the slice at $t = -\infty$ (in-state) to the slice at $t = +\infty$ (out-state). The action by the operator $\Gamma_-(1)$ allows to generate all possible 2d- partitions on a given diagonal slice as shown on eq(2.22). The relation (2.16) permits to move from a slice to an other by creating all possible partitions interlacing with the partitions in the previous slice.

Therefore, using eq(2.16) and $q^{L_0}|0\rangle = |0\rangle$, the states (3.11-3.12) can be rewritten as

$$\Psi_-(1)|0\rangle = \left(\prod_{k=0}^{\infty} \Gamma_-(q^k) \right) |0\rangle, \quad (3.13)$$

and similar relation for $\langle 0|\Psi_+(1)$. We deduce from this relation the two following:

(i) 3d partitions can be realized in terms of an infinite 2d ones.

²Note that $\Psi_{\pm}(1)$ corresponds to $\Psi_{\pm}(z)$ with $z = 1$. Note also that $\Psi_{\pm}(1)$ depend on the q-parameter; it has been dropped out for simplicity of notations.

(ii) the generating functional of 3d partitions are captured by the local vertex operators³

$$\Psi_{-}(1) = \lim_{s \rightarrow \infty} \left(\prod_{k=0}^s \Gamma_{-}(q^k) \right) q^{sL_0}, \quad (3.14)$$

and its dual $\Psi_{+}(1)$. These Ψ_{\pm} operators will be denoted later as

$$\Psi_{\pm}(z) = \Gamma_{\pm}^{(2)}(z), \quad (3.15)$$

but to keep the notations simpler, we will momentary use $\Psi_{\pm}(z)$ and come later to the $\Gamma_{\pm}^{(2)}$ when we consider *p-dimensional* generalization.

3.2 More on 3d generating function

The partition function Z_{3d} generating 3d- generalized Young diagrams is given, in the transfer matrix language, as follows

$$Z_{3d} = \left\langle 0 \left| \left(\prod_{t=0}^{\infty} q^{L_0} \Gamma_{+}(1) \right) q^{L_0} \left(\prod_{t=-\infty}^{-1} \Gamma_{-}(1) q^{L_0} \right) \right| 0 \right\rangle. \quad (3.16)$$

Splitting q^{L_0} as $q^{\frac{L_0}{2}} q^{\frac{L_0}{2}}$ and commuting each of the operators $q^{\frac{L_0}{2}}$ to the left and the other to the right by using eq(2.15), we get

$$Z_{3d} = \left\langle 0 \left| \prod_{t=0}^{\infty} \Gamma_{+}\left(q^{-t-\frac{1}{2}}\right) \prod_{l=1}^{\infty} \Gamma_{-}\left(q^{l-\frac{1}{2}}\right) \right| 0 \right\rangle. \quad (3.17)$$

Then commuting the Γ_{-} 's to the left of Γ_{+} , we obtain

$$Z_{3d} = \left(\prod_{l=0}^{\infty} \left[\prod_{j=1}^{\infty} \left(\frac{1}{(1 - q^{j+l})} \right) \right] \right), \quad (3.18)$$

By setting $j + l = k$, we can bring this relation to

$$Z_{3d} = \left(\prod_{k=1}^{\infty} \left[\prod_{j=1}^k \left(\frac{1}{(1 - q^k)} \right) \right] \right), \quad (3.19)$$

and then to

$$Z_{3d} = \prod_{k=1}^{\infty} \left(\frac{1}{(1 - q^k)^k} \right), \quad (3.20)$$

which is precisely the usual form of the 3d- MacMahon function. Before proceeding ahead notice the four following:

³It should be noted that a 3d partition is a collection of Young diagrams. However, an arbitrary collection of Young diagrams do not correspond to a 3d partition.

(1) Z_{3d} as a *free* CFT₂ with central charge $c \rightarrow \infty$

The expression (3.18) of Z_{3d} is very suggestive. Setting

$$Z_{2d}^{(l)} = \prod_{j=1}^{\infty} \left(\frac{1}{(1 - Q_l q^j)} \right) , \quad Q_l = q^l , \quad (3.21)$$

which, roughly, describes a partition function Z_{2d} , we could then rewrite eq(3.18) like

$$Z_{3d} \sim \prod_{l=0}^{\infty} Z_{2d}^{(l)} . \quad (3.22)$$

Seen that each $Z_{2d}^{(l)}$ is associated with a $c = 1$ *free* CFT₂ representation, it follows that Z_{3d} could be interpreted as the partition function of a *free* CFT₂ representation with $c \rightarrow \infty$. In section 6, we develop an alternative interpretation of Z_{3d} using correlation of $c = 1$ *q- deformed* vertex operators.

(2) Vertex operators $\Psi_{\pm}(z)$: *Level 2*.

Using eqs(2.11-3.11), it is not difficult to check that $\Psi_{\pm}(z)$ is also a local vertex operator whose explicit expression in terms of the $J_{\pm n}$ modes, reads as,

$$\begin{aligned} \Psi_{-}(z) &= \exp \left(i \sum_{n>0} \frac{1}{n} \left(\frac{z^n}{1 - q^n} \right) J_{-n} \right) , \\ \Psi_{+}(z) &= \exp \left(-i \sum_{n>0} \frac{1}{n} \left(\frac{z^{-n}}{1 - q^n} \right) J_n \right) . \end{aligned} \quad (3.23)$$

The explicit derivation of these relations is given in appendix A, eq(8.2). Notice that:

(i) $\Gamma_{\pm}(z)$ and $\Psi_{\pm}(z)$ are related by the mapping

$$z^{\pm n} \rightarrow \frac{z^{\pm n}}{1 - q^n} , \quad (3.24)$$

Since for $q \rightarrow 0$, $\Gamma_{\pm}(z)$ and $\Psi_{\pm}(z)$ coincide, it follows that the operators $\Psi_{\pm}(z)$ can be interpreted as a *q- deformation* $\Gamma_{\pm}(z)$.

(ii) $\Gamma_{\pm}(z)$ and $\Psi_{\pm}(z)$ share most of the basic quantum properties since both of them involve the same Kac-Moody mode operators $J_{\pm n}$,

(3) Translations

The operator q^{L_0} acts also as a translation operator on $\Psi_{\pm}(z)$ in the same manner like for $\Gamma_{\pm}(z)$.

$$q^{L_0} \Psi_{\pm}(z) q^{-L_0} = \Psi_{\pm}(qz) , \quad (3.25)$$

This property allows us to define 4d- generalization from 3d one in quite similar manner as we have done in going from 2d to 3d. We will come back to this feature later.

(4) Z_{3d} as a "2- point correlation" function.

Using $\Psi_{\pm}(1)$ vertex operators, the partition function Z_{3d} can be put in the simplest form

$$Z_{3d} = \langle 0 | \Psi_{+}(1) q^{L_0} \Psi_{-}(1) | 0 \rangle . \quad (3.26)$$

By help of the identity $q^{L_0}\Psi_-(1)q^{-L_0} = \Psi_-(q)$ eq(3.25), we also have

$$Z_{3d} = \langle 0 | \Psi_+(1) \Psi_-(q) | 0 \rangle , \quad (3.27)$$

where Z_{3d} appears as just the 2- point correlation function of the *level 2* vertex operators $\Psi_+(1)$ and $\Psi_-(q)$. It happens that eq(3.27) is not the unique way to define Z_{3d} . Let us comment briefly aspects of this issue; general results will be given in sections 5 and 6.

(i) Eq(3.27) can be also expressed as follows

$$Z_{3d} = \left\langle 0 | \Gamma_+(1) q^{L_0} \left(\prod_{t=-\infty}^{-1} \Psi_-(1) q^{L_0} \right) | 0 \right\rangle . \quad (3.28)$$

This expression will allow us to get the definition of higher dimensional generalizations of MacMahon function; see eq(5.2). This relation can be put in the simple form

$$Z_{3d} = \langle 0 | \Gamma_+(1) q^{L_0} \Omega_-(1) | 0 \rangle , \quad (3.29)$$

or equivalently

$$Z_{3d} = \langle 0 | \Gamma_+(1) \Omega_-(q) | 0 \rangle , \quad (3.30)$$

where we have set

$$\Omega_-(1) = \lim_{s \rightarrow \infty} \left(\prod_{t=0}^s \Psi_-(q^k) \right) q^{sL_0} , \quad (3.31)$$

This local vertex operator should be thought of as the *level 3* of the hierarchy we have refereed to earlier; i.e.

$$\Omega_{\pm}(1) = \Gamma_{\pm}^{(3)} , \quad (3.32)$$

(ii) Along with the two representations (3.27) and (3.30) the partition function Z_{3d} can be expressed as well like

$$Z_{3d} = \left\langle 0 | \Omega_+ \left(\frac{1}{q} \right) \Gamma_-(1) | 0 \right\rangle , \quad (3.33)$$

where we have used the correlation of $\Omega_+(x)$ and $\Gamma_-(y)$ rather than $\Gamma_+(x)$ and $\Omega_-(y)$.

(iii) The diversity in expressing Z_{3d} as a 2- point correlation function, let us suspect that Z_{3d} could be expressed as a more basic objects. In exploring this idea, we have found that the adequate interpretation of $Z_{3d}(q)$ is as a special 4- point correlation function

$$Z_{3d}(q) = \mathcal{G}_4(x_0, x_1, x_2, x_3) , \quad (3.34)$$

of vertex operators $\mathcal{O}_j(x_j)$ involving different $\Gamma_{\pm}^{(p)}$ levels,

$$\mathcal{G}_4 = \langle 0 | \mathcal{O}_0(x_0) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) | 0 \rangle . \quad (3.35)$$

To fix the ideas keep in mind the two following:

(α) the vertex operator $\mathcal{O}_0(x_0)$ stands for $\Gamma_+(1)$ and the other operators will be explicitly given in section 6; see eqs (6.34).

(β) the observed diversity in defining $Z_{3d}(q)$ corresponds just to decomposing (3.35) by using Wick theorem combined with eqs(2.18-2.19).

4 Extension to 4d and 5d

We first show that the leading terms of the generalized MacMahon function can be realized as 2- point functions of some vertex operators of $c = 1$ 2d conformal field theory. Then, we use this feature to derive the general formula for $G_d(q)$. In section 5, we consider the interpretation of $G_d(q)$ as $(d + 1)$ - points correlation function $\mathcal{G}_d(x_0, x_1, \dots, x_d)$ involving vertex operators $\mathcal{O}_j(x_j)$ as in eq(1.2).

4.1 Z_{1d} and Z_{2d} as 2-point functions

Before studying 4d and 5d generalizations, it is interesting to start by revisiting the 1d and 2d cases. This is an important thing for getting the full picture on the conjectured MacMahon function G_d .

We start by noting the two following:

(1) Recall that the 1d- MacMahon function corresponds to,

$$Z_{1d} = \frac{1}{1 - q}. \quad (4.1)$$

This function can be *exactly* interpreted as the two- point correlation⁴

$$Z_{1d} = \mathcal{G}_2 = \mathcal{G}_2(z_0, z_1), \quad (4.2)$$

of the vertex operators $\Gamma_+(1)$ and $\Gamma_-(q)$ as shown below,

$$Z_{1d} = \langle 0 | \Gamma_+(1) \Gamma_-(q) | 0 \rangle. \quad (4.3)$$

This relation describes just the bosonization of eq(2.5) and can be rewritten, by using the hamiltonian L_0 , as follows,

$$Z_{1d} = \langle 0 | \Gamma_+(1) q^{L_0} \Gamma_-(1) | 0 \rangle \quad (4.4)$$

(2) A quite similar interpretation can be also given for 2d- MacMahon function,

$$Z_{2d} = \prod_{k \geq 1} (1 - q^k)^{-1} \quad (4.5)$$

This relation can be expressed in terms of operators vertex as follows,

$$Z_{2d} = \left\langle 0 | \Gamma_+(1) q^{L_0} \left(\prod_{k \geq 1} \Gamma_-(1) q^{L_0} \right) | 0 \right\rangle, \quad (4.6)$$

Indeed using (2.16), we can bring it to

$$Z_{2d} = \langle 0 | \Gamma_+(1) \left(\prod_{k \geq 1} \Gamma_-(q^k) \right) | 0 \rangle \quad (4.7)$$

⁴ Z_{pd} is the MacMahon function G_{pd} ; it shouldn't be confused with its interpretation as $(p + 1)$ - points correlation function $\mathcal{G}_{p+1} = \mathcal{G}(x_0, x_1, \dots, x_p)$ to be studied in section 6; see also eqs(1.1).

Then moving the operators $\Gamma_- (q^k)$ to the left and $\Gamma_+ (1)$ to the right by using eqs(2.15), we get the desired result.

Notice that using eq(3.11), we learn that Z_{2d} can be also defined as two- point correlation function as follows

$$Z_{2d} = \langle 0 | \Gamma_+ (1) \Psi_- (q) | 0 \rangle . \quad (4.8)$$

This relation involves the correlation of two vertex operators of different levels namely Γ_+ (*level 1*) and $\Psi_- (q)$ (*level 2*). Remark also that though Γ_+ and Γ_- do not appear on equal footing in eq(4.6-4.8), positivity of Z_{2d} is ensured because Γ_+ and Γ_- are positive defined operators. Notice moreover that we also have

$$Z_{2d} = \mathcal{G}_3 = \mathcal{G}_3 (z_0, z_1, z_3) , \quad (4.9)$$

but this feature will be discussed later on once we give the derivation proof of the conjectured MacMahon function G_d .

4.2 Z_{pd} derivation for $p = 4, 5$

The property that Z_{1d} (4.3), Z_{2d} (4.8) and Z_{3d} (3.27) can be all of them interpreted as 2- point correlation functions of some given vertex operators is very remarkable. It happens in fact that this feature is a more general property valid also for higher dimensional generalizations. Let us describe this feature here for the 4d and 5d cases.

Motivated by the above analysis, 4d- and 5d- generalizations of the MacMahon function can be then defined as well as 2- point correlation functions of some local operators as follows

$$\begin{aligned} Z_{4d} &= \langle 0 | \Psi_+ (1) \Omega_- (q) | 0 \rangle , \\ Z_{5d} &= \langle 0 | \Omega_+ (1) \Omega_- (q) | 0 \rangle , \end{aligned} \quad (4.10)$$

where $\Omega_{\pm} (q)$ are vertex operators of some hierarchy level (*level 3*) which remain to be specified. These relations have been motivated by the following,

$$\begin{aligned} Z_{2d} &= \langle 0 | \Gamma_+ (1) \Psi_- (q) | 0 \rangle , \\ Z_{3d} &= \langle 0 | \Psi_+ (1) \Psi_- (q) | 0 \rangle , \end{aligned} \quad (4.11)$$

and also

$$\begin{aligned} Z_{0d} &= \langle 0 | I_{id} (1) \Gamma_- (q) | 0 \rangle , \\ Z_{1d} &= \langle 0 | \Gamma_+ (1) \Gamma_- (q) | 0 \rangle , \end{aligned} \quad (4.12)$$

where I_{id} stands for the identity operator (of level zero). To get the $\Omega_{\pm} (z)$ operators, we require:

(i) The $\Omega_+(z)$ and $\Omega_-(z)$ are local CFT₂ vertex operators that should obey

$$\begin{aligned}\Omega_-(x)\Omega_-(y) &= \Omega_-(y)\Omega_-(x), \\ \Omega_-(x)\Psi_-(y) &= \Psi_-(y)\Omega_-(x), \\ \Omega_-(x)\Gamma_-(y) &= \Gamma_-(y)\Omega_-(x), \\ \Omega_-(0) &= 1,\end{aligned}\tag{4.13}$$

and similar relations for $\Omega_+(x)$.

(ii) We should also have

$$\Omega_-(q) = q^{L_0}\Omega_-(1)q^{-L_0}\tag{4.14}$$

so that

$$\begin{aligned}Z_{4d} &= \langle 0|\Psi_+(1)q^{L_0}\Omega_-(1)|0\rangle, \\ Z_{5d} &= \langle 0|\Omega_+(1)q^{L_0}\Omega_-(1)|0\rangle,\end{aligned}\tag{4.15}$$

in analogy with the transfer matrix method used previously.

(iii) We impose the commutation relations

$$\begin{aligned}\Psi_+(1)\Omega_-(q) &= G_4(q)\Omega_-(q)\Psi_+(1), \\ \Omega_+(1)\Omega_-(q) &= G_5(q)\Omega_-(q)\Omega_+(1),\end{aligned}\tag{4.16}$$

where $G_4(q)$ and $G_5(q)$ stand for the 4d- and 5d- generalized MacMahon functions given by,

$$\begin{aligned}G_4(q) &= \prod_{k=1}^{\infty} \left[\left(\frac{1}{1-q^k} \right)^{\frac{(k+1)!}{(k-1)!2!}} \right], \\ G_5(q) &= \prod_{k=1}^{\infty} \left[\left(\frac{1}{1-q^k} \right)^{\frac{(k+2)!}{(k-1)!3!}} \right].\end{aligned}\tag{4.17}$$

A solution of these constraint relations is given by

$$\begin{aligned}\Omega_-(1) &= \left(\prod_{t_2=-\infty}^{-1} \left(\prod_{t_1=-\infty}^{-1} (\Gamma_-(1)q^{L_0}) \right) q^{L_0} \right), \\ \Omega_+(1) &= \left(\prod_{t_2=0}^{\infty} q^{L_0} \left(\prod_{t_1=0}^{\infty} (q^{L_0}\Gamma_+(1)) \right) \right),\end{aligned}\tag{4.18}$$

or equivalently like

$$\begin{aligned}\Omega_-(1) &= \left(\prod_{t=-\infty}^{-1} (\Psi_-(1)) q^{L_0} \right), \\ \Omega_+(1) &= \left(\prod_{t=0}^{\infty} q^{L_0}\Psi_+(1) \right).\end{aligned}\tag{4.19}$$

To check that these relations solve indeed the above constraint eqs, let us give some explicit details.

4d case:

First consider the 4d- partition function Z_{4d} expressed in (4.15) which we rewrite by substituting (4.19) as follows,

$$Z_{4d} = \left\langle 0 | \Psi_+(1) q^{L_0} \left(\prod_{t=-\infty}^{-1} \Psi_-(1) q^{L_0} \right) | 0 \right\rangle. \quad (4.20)$$

By help of eq(4.14), it reads also like

$$Z_{4d} = \left\langle 0 | \Psi_+(1) \left(\prod_{l=1}^{\infty} \Psi_-(q^l) \right) | 0 \right\rangle. \quad (4.21)$$

Then commuting $\Psi_-(q^l)$ to the left by using the identity

$$\Psi_+(1) \Psi_-(x) = \left[\prod_{k=1}^{\infty} \left(\frac{1}{(1 - xq^{k-1})^k} \right) \right] \Psi_-(x) \Psi_+(1), \quad x < 1, \quad (4.22)$$

see also appendix A eqs(8.6) for general case, we get

$$\begin{aligned} Z_{4d} &= \prod_{k=1}^{\infty} \prod_{l=1}^{\infty} \left(\frac{1}{(1 - q^{l+k-1})^k} \right) \\ &= \prod_{s=1}^{\infty} \prod_{k=1}^s \left(\frac{1}{(1 - q^s)^k} \right) \end{aligned} \quad (4.23)$$

which, up on using $\sum_{k=1}^s k = \frac{s(s+1)}{2}$, can be also put in the form

$$Z_{4d} = \prod_{s=1}^{\infty} \left(\frac{1}{(1 - q^s)^{\frac{s(s+1)}{2}}} \right), \quad (4.24)$$

that should be compared with eq(4.17). Notice that like for Z_{3d} , the 4d partition function can be expressed in different, but equivalent, ways: We have the results

$$Z_{4d} = \begin{cases} \langle 0 | \Psi_+(1) \Omega_-(q) | 0 \rangle, \\ \langle 0 | \Omega_+\left(\frac{1}{q}\right) \Psi_-(1) | 0 \rangle, \\ \langle 0 | \Upsilon_+(1) \Upsilon_-(q) | 0 \rangle, \\ \langle 0 | \Upsilon_+\left(\frac{1}{q}\right) \Gamma_-(1) | 0 \rangle, \end{cases} \quad (4.25)$$

where we have set

$$\Upsilon_-(q) = \left(\prod_{k=1}^{\infty} \Omega_-(q^k) \right) \quad (4.26)$$

which should be thought of as $\Upsilon_- = \Gamma_-^{(4)}$.

5d case

Similarly, we have for the 5d- generalization (4.15),

$$Z_{5d} = \left\langle 0 \left| \left(\prod_{t=0}^{\infty} q^{L_0} \Psi_+(1) \right) q^{L_0} \Omega_-(1) \right| 0 \right\rangle, \quad (4.27)$$

where we have substituted Ω_+ in terms of product of Ψ_+ (4.19). Next using the fact that q^{L_0} acts as a translation operator, we can put Z_{5d} as follows

$$Z_{5d} = \left\langle 0 \left| \left(\prod_{l=1}^{\infty} \Psi_+(q^{-l}) \right) \Omega_-(1) \right| 0 \right\rangle. \quad (4.28)$$

Then using the identity

$$\Psi_+\left(\frac{1}{x}\right)\Omega_-(1) = \left[\prod_{s=1}^{\infty} \left(\frac{1}{1-xq^s} \right)^{\frac{s(s+1)}{2}} \right] \Omega_-(1) \Psi_+\left(\frac{1}{x}\right), \quad (4.29)$$

we obtain

$$Z_{5d} = \prod_{s=1}^{\infty} \prod_{l=1}^{\infty} \left[\left(\frac{1}{1-q^{l+s}} \right)^{\frac{s(s+1)}{2}} \right]. \quad (4.30)$$

The next step is to put it in the form

$$Z_{5d} = \prod_{k=1}^{\infty} \prod_{s=1}^k \left[\left(\frac{1}{1-q^k} \right)^{\frac{s(s+1)}{2}} \right], \quad (4.31)$$

which gives

$$Z_{5d} = \prod_{k=1}^{\infty} \left[\left(\frac{1}{1-q^k} \right)^{\frac{k(k+1)(k+2)}{6}} \right]. \quad (4.32)$$

In getting this relation, we have used the identity

$$\sum_{s=1}^k \frac{s(s+1)}{2} = \frac{k(k+1)(k+2)}{6}, \quad (4.33)$$

proved in appendix B. Here also we have different, but equivalent, ways to define Z_{5d} . Later on, we will give the exact numbers of ways for generic Z_{pd} .

5 Result by induction

First notice that the expression (3.27) of the partition function Z_{3d} can be also put in the form

$$Z_{3d} = \left\langle 0 \left| \Gamma_+(1) q^{L_0} \left(\prod_{t_2=-\infty}^{-1} (\Psi_-(1) q^{L_0}) \right) \right| 0 \right\rangle \quad (5.1)$$

or equivalently by using eq(3.11), like

$$Z_{3d} = \left\langle 0 \left| \Gamma_+(1) q^{L_0} \left(\prod_{t_2=-\infty}^{-1} \left[\left(\prod_{t_1=-\infty}^{-1} (\Gamma_-(q) q^{L_0}) \right) q^{L_0} \right] \right) \right| 0 \right\rangle. \quad (5.2)$$

This relation as well as eqs(4.4-4.6) suggest us the structure of the p -dimensional partition function Z_{pd} in terms of CFT₂'s vertex operators Γ_{\pm} . For doing so, we need to introduce the following hierarchy of local vertex operators

$$\Gamma_{-}^{(n+1)}(1) = \left(\prod_{t_n=-\infty}^{-1} \cdots \prod_{t_2=-\infty}^{-1} \left[\left(\prod_{t_1=-\infty}^{-1} (\Gamma_{-}(1) q^{L_0}) \right) q^{L_0} \right] \cdots q^{L_0} \right) \quad (5.3)$$

for $n \geq 1$, together with

$$\Gamma_{-}^{(0)} = I_{id}, \quad \Gamma_{-}^{(1)}(z) = \Gamma_{-}(z). \quad (5.4)$$

Eq(5.3) can be also defined as follows,

$$\Gamma_{-}^{(n+1)}(1) = \prod_{t=-\infty}^{-1} \left(\Gamma_{-}^{(n)}(1) q^{L_0} \right), \quad n \geq 1. \quad (5.5)$$

A similar relation can be written down for $\Gamma_{+}^{(n+1)}(1)$. The $\Gamma_{-}^{(p)}$, referred to as the *level p* vertex operator, obey quite similar relations that the ones associated to $\Gamma_{-}(1)$, in particular

$$\Gamma_{-}^{(p)}(q) = q^{L_0} \Gamma_{-}^{(p)}(1) q^{-L_0}, \quad p \geq 0. \quad (5.6)$$

More details, concerning these high level operators, are presented in Appendix A.

Based on the preceding results realized for lower dimensions, it follows that the p -dimensional partition functions Z_{pd} can be defined as,

$$Z_{pd} = \left\langle 0 | \Gamma_{+}(1) \Gamma_{-}^{(p)}(q) | 0 \right\rangle, \quad p \geq 0. \quad (5.7)$$

This relation, which has been explicitly checked for $p = 0, 1, 2, 3, 4$ and 5, reads also as

$$Z_{pd} = \left\langle 0 | \Gamma_{+}(1) q^{L_0} \Gamma_{-}^{(p)}(1) | 0 \right\rangle. \quad (5.8)$$

Commuting $\Gamma_{-}^{(p)}(q)$ to the left of $\Gamma_{+}(1)$, we can show by induction that for $p \geq 2$

$$Z_{pd} = \prod_{k=1}^{\infty} \left[\left(\frac{1}{1 - q^k} \right)^{\frac{(k+p-3)!}{(k-1)!(p-2)!}} \right], \quad (5.9)$$

Proof by induction:

We suppose that eq(5.9) holds for *level p* ; then prove that it holds as well for *level $(p+1)$* ; that is,

$$Z_{(p+1)d} = \left\langle 0 | \Gamma_{+}(1) \Gamma_{-}^{(p+1)}(q) | 0 \right\rangle, \quad (5.10)$$

and find that it is given by

$$Z_{(p+1)d} = \prod_{k=1}^{\infty} \left[\left(\frac{1}{1 - q^k} \right)^{\frac{(k+p-2)!}{(k-1)!(p-1)!}} \right]. \quad (5.11)$$

Indeed, we start from the definition of $Z_{(p+1)d}$,

$$Z_{(p+1)d} = \left\langle 0 | \Gamma_+ (1) \Gamma_-^{(p+1)} (q) | 0 \right\rangle \quad (5.12)$$

and express it, by using (5.5), as

$$Z_{(p+1)d} = \left\langle 0 | \Gamma_+ (1) q^{L_0} \left(\prod_{t=-\infty}^{-1} \Gamma_-^{(p)} (1) q^{L_0} \right) | 0 \right\rangle \quad (5.13)$$

or equivalently like

$$Z_{(p+1)d} = \left\langle 0 | \Gamma_+ (1) \left(\prod_{l=1}^{\infty} \Gamma_-^{(p)} (q^l) \right) | 0 \right\rangle. \quad (5.14)$$

Then we commute $\Gamma_-^{(p)} (q^l)$ to the left of $\Gamma_+ (1)$ in eq(5.10), we get after some computations,

$$Z_{(p+1)d} = \prod_{l=1}^{\infty} \prod_{k=1}^{\infty} \left[\left(\frac{1}{1 - q^{l+k}} \right)^{\frac{(k+p-3)!}{(k-1)!(p-2)!}} \right]. \quad (5.15)$$

Setting $s = (l + k)$, we can rewrite this relation as follows

$$Z_{(p+1)d} = \prod_{s=1}^{\infty} \prod_{k=1}^s \left[\left(\frac{1}{1 - q^s} \right)^{\frac{(k+p-3)!}{(k-1)!(p-2)!}} \right]. \quad (5.16)$$

At first sight, this expression seems different from the desired result; however explicit computation leads exactly to the right result; thanks to the combinatorial identity,

$$\sum_{k=1}^s \frac{(k+p-3)!}{(k-1)!(p-2)!} = \frac{(s+p-2)!}{(s-1)!(p-1)!}, \quad p \geq 2, \quad (5.17)$$

which is showed in appendix B.

These computations give an explicit proof for the derivation of the expression of generalized MacMahon function. Thanks to the "transfer matrix method" and to the hierarchy of *level p* vertex operators $\Gamma_{\pm}^{(p)}$ eq(5.5).

6 $G_n(q)$ as $(n+1)$ -point correlation function

So far we have seen that n - dimensional generalization of MacMahon function $G_n(q)$ with $n \geq 2$, can be interpreted as 2- point correlation functions of some composite vertex operators. We have also seen that there are different, but equivalent ways, to express $G_n(q)$ as 2- point correlation functions. Using $\Gamma_{\pm}^{(r)}$ and $\Gamma_{\pm}^{(s)}$ vertex operators, one can check that for any positive definite integers r and s such that $r + s - 1 = n$, we have,

$$G_n(q) = \langle 0 | \Gamma_+^{(n-s+1)} (1) \Gamma_-^{(s)} (q) | 0 \rangle, \quad 1 \leq s \leq n. \quad n \geq 1. \quad (6.1)$$

The $2 \binom{r+s-2}{2} + 1$ possibilities are all of them equal to each other. This diversity in defining $G_n(q)$ suggests us to look for a more refined definition of it. We have found

that the adequate way to define $G_n(q)$ is like a $(n+1)$ - point correlation function as given below,

$$G_n(q) = \mathcal{G}_{n+1}(x_0, x_1, x_2 \cdots, x_n) \quad (6.2)$$

with

$$\mathcal{G}_{n+1} = \langle 0 | \mathcal{O}_0(x_0) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n) | 0 \rangle, \quad (6.3)$$

where the $x_j = x_j(q)$ and $\mathcal{O}_j(x_j)$ are some vertex operators that have to be specified. In this way, the diversity (6.1) appears just as a manifestation of applying Wick theorem to (6.3) for its decomposition in terms of two- points correlation functions. To fix the ideas, think about $\mathcal{O}_0(x_0)$ as given by

$$\mathcal{O}_0(x_0) = \Gamma_+(1) \quad (6.4)$$

and all remaining others as given by vertex operators involving products of $\Gamma_-(y)$ only, that is:

$$\mathcal{O}_j(x_j) \sim \prod \Gamma_-(y), \quad j = 1, \dots, n. \quad (6.5)$$

Since $\Gamma_-(y)$ and any product of $\Gamma_-(y)$ has vacuum expectation values equal to one,

$$\langle 0 | \Gamma_-(y) | 0 \rangle = 1 = \left\langle 0 \left| \left(\prod \Gamma_-(y) \right) \right| 0 \right\rangle, \quad (6.6)$$

it follows that

$$\left\langle 0 \left| \prod_{l=1}^n \mathcal{O}_l(x_l) \right| 0 \right\rangle = 1. \quad (6.7)$$

Then, by using Wick theorem \mathcal{G}_{n+1} reduces to

$$\mathcal{G}_{n+1} = \prod_k \langle 0 | \mathcal{O}_0(x_0) \mathcal{O}_k(x_k) | 0 \rangle, \quad k = 1, \dots, n. \quad (6.8)$$

Let us build the Green function \mathcal{G}_{n+1} step by step, starting from eq(5.7) and using the results obtained above:

6.1 Leading terms

Below, we give the explicit computation of \mathcal{G}_n for $n = 2, 3, 4, 5$

(1) $G_1(q)$ as 2- point propagator.

Comparing

$$G_1(q) = \langle 0 | \Gamma_+(1) \Gamma_-(q) | 0 \rangle \quad (6.9)$$

with

$$\mathcal{G}_2 = \langle 0 | \mathcal{O}_0(x_0) \mathcal{O}_1(x_1) | 0 \rangle \quad (6.10)$$

we get

$$\begin{aligned} \mathcal{O}_0(x_0) &= \Gamma_+(1), \\ \mathcal{O}_1(x_1) &= \Gamma_-(q) \end{aligned} \quad (6.11)$$

(2) $G_2(q)$ as a 3-point function

Starting from the expression eq(5.7) for G_2 ,

$$G_2(q) = \langle 0 | \Gamma_+(1) \Gamma_-^{(2)}(q) | 0 \rangle, \quad (6.12)$$

and using the special property established in appendix A; see eq(8.26),

$$\Gamma_-^{(2)}(q) = \Gamma_-^{(1)}(q) \Gamma_-^{(2)}(q^2) \quad (6.13)$$

we can bring $G_2(q)$ into the form

$$G_2(q) = \langle 0 | \Gamma_+(1) \Gamma_-^{(1)}(q) \Gamma_-^{(2)}(q^2) | 0 \rangle \quad (6.14)$$

Then, comparing with

$$\mathcal{G}_3 = \langle 0 | \mathcal{O}_0(x_0) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) | 0 \rangle, \quad (6.15)$$

we get, in addition to eqs(6.11), the following:

$$\mathcal{O}_2(x_2) = \Gamma_+^{(2)}(q^2). \quad (6.16)$$

(3) $G_3(q)$ as a 4-point function

We start from the expression of $G_3(q)$,

$$G_3(q) = \langle 0 | \Gamma_+(1) \Gamma_-^{(3)}(q) | 0 \rangle \quad (6.17)$$

then use the identity,

$$\Gamma_-^{(3)}(q) = \Gamma_-^{(2)}(q) \Gamma_-^{(3)}(q^2), \quad (6.18)$$

and substitute $\Gamma_-^{(2)}(q)$ by eq(6.13), we get

$$\Gamma_-^{(3)}(q) = \Gamma_-^{(1)}(q) \left(\prod_{l=1}^2 \Gamma_-^{(2)}(q^2) \right) \Gamma_-^{(3)}(q^3). \quad (6.19)$$

Comparing with

$$\mathcal{G}_3 = \langle 0 | \mathcal{O}_0(x_0) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) | 0 \rangle, \quad (6.20)$$

we obtain

$$\begin{aligned} \mathcal{O}_0(x_0) &= \Gamma_+(1) \\ \mathcal{O}_1(x_1) &= \Gamma_-^{(1)}(q) \\ \mathcal{O}_2(x_2) &= \left(\prod_{l=1}^2 \Gamma_-^{(2)}(q^2) \right) \\ \mathcal{O}_3(x_3) &= \Gamma_-^{(3)}(q^3) \end{aligned} \quad (6.21)$$

(4) $G_4(q)$ as a 5-point function

Starting from

$$G_4(q) = \langle 0 | \Gamma_+(1) \Gamma_-^{(4)}(q) | 0 \rangle \quad (6.22)$$

then using the identities,

$$\begin{aligned}\Gamma_-^{(4)}(q) &= \Gamma_-^{(3)}(q) \Gamma_-^{(4)}(q^2), \\ \Gamma_-^{(4)}(q^2) &= \Gamma_-^{(3)}(q^2) \Gamma_-^{(4)}(q^3) \\ \Gamma_-^{(4)}(q^3) &= \Gamma_-^{(3)}(q^3) \Gamma_-^{(4)}(q^4)\end{aligned}\tag{6.23}$$

we obtain at a first stage

$$\Gamma_-^{(4)}(q) = \Gamma_-^{(3)}(q) \Gamma_-^{(3)}(q^2) \Gamma_-^{(3)}(q^3) \Gamma_-^{(4)}(q^4).\tag{6.24}$$

At a second stage, we substitute $\Gamma_-^{(3)}(q)$ and $\Gamma_-^{(3)}(q^2)$ by eq(6.18), we get

$$\begin{aligned}\Gamma_-^{(3)}(q) &= \Gamma_-^{(2)}(q) \Gamma_-^{(3)}(q^2), \\ \Gamma_-^{(3)}(q^2) &= \Gamma_-^{(2)}(q^2) \Gamma_-^{(3)}(q^3).\end{aligned}\tag{6.25}$$

Putting back into eq(6.24), we obtain

$$\Gamma_-^{(4)}(q) = \Gamma_-^{(2)}(q) \Gamma_-^{(2)}(q^2) \Gamma_-^{(2)}(q^2) \Gamma_-^{(3)}(q^3) \Gamma_-^{(3)}(q^3) \Gamma_-^{(3)}(q^3) \Gamma_-^{(4)}(q^4).$$

Next replacing $\Gamma_-^{(2)}(q)$ by eq(6.13), we end with the following,

$$\Gamma_-^{(4)}(q) = \Gamma_-^{(1)}(q) \left(\prod_{l=1}^3 \Gamma_-^{(2)}(q^2) \right) \left(\prod_{l=1}^3 \Gamma_-^{(3)}(q^3) \right) \Gamma_-^{(4)}(q^4).\tag{6.26}$$

Comparing with

$$\mathcal{G}_5 = \langle 0 | \mathcal{O}_0(x_0) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) | 0 \rangle,\tag{6.27}$$

we obtain

$$\begin{aligned}\mathcal{O}_0(x_0) &= \Gamma_+(1), \\ \mathcal{O}_1(x_1) &= \Gamma_-^{(1)}(q), \\ \mathcal{O}_2(x_2) &= \left(\prod_{l=1}^3 \Gamma_-^{(2)}(q^2) \right), \\ \mathcal{O}_3(x_3) &= \left(\prod_{l=1}^3 \Gamma_-^{(3)}(q^3) \right), \\ \mathcal{O}_4(x_4) &= \Gamma_-^{(4)}(q^4).\end{aligned}\tag{6.28}$$

With these results on lower values of p , it is straightforward to derive the generic picture.

6.2 Generic result

We start from the expression of

$$G_p(q) = \left\langle 0 | \Gamma_+(1) \Gamma_-^{(p)}(q) | 0 \right\rangle, \quad p \geq 1.\tag{6.29}$$

Then we use the identity,

$$\Gamma_-^{(p)}(q) = \prod_{k=0}^{p-1} \left(\prod_{l_k=1}^{p_k} \Gamma_-^{(k+1)}(q^{k+1}) \right), \quad (6.30)$$

with

$$p_k = \frac{(p-1)!}{k!(p-k-1)!}, \quad 0 \leq k \leq p-1, \quad (6.31)$$

which is proved in appendix A, eq(8.37), we can bring $G_p(q)$ into the form,

$$G_p(q) = \left\langle 0 | \Gamma_+(1) \prod_{k=0}^{p-1} \left(\prod_{l_k=1}^{p_k} \Gamma_-^{(k+1)}(q^{k+1}) \right) | 0 \right\rangle, \quad p \geq 1. \quad (6.32)$$

By comparing with,

$$\mathcal{G}_{p+1} = \langle 0 | \mathcal{O}_0(x_0) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_p(x_p) | 0 \rangle, \quad (6.33)$$

we obtain

$$\begin{aligned} \mathcal{O}_0(x_0) &= \Gamma_+(1), & \mathcal{O}_1(x_1) &= \Gamma_-^{(1)}(q), \\ \mathcal{O}_j(x_j) &= \left(\prod_{l_j=1}^{p_j} \left[\Gamma_-^{(j)}(q^j) \right] \right), & j &\geq 2. \end{aligned} \quad (6.34)$$

For more details on the derivation of this relation, see appendix A: eqs(8.27)-(8.37). Eqs(6.34) complete the interpretation of G_d as a $(d+1)$ - points Green function.

7 Discussion and Conclusion

In this paper, we have given a 2d- conformal field theoretical derivation of the generalized MacMahon function by using ideas from "transfer matrix method" and q-deformed QFT₂. Among our results, we mention:

(1) The usual vertex operators $\Gamma_{\pm}(z)$ of the bosonic $c = 1$ conformal field theory appear as the *level one* of the following hierarchy,

$$\Gamma_-^{(p)}(z) | 0 \rangle = \exp \left(\sum_{n=1}^{\infty} \frac{iz^n J_{-n}}{n(1-q^n)^{p-1}} \right) | 0 \rangle, \quad p \geq 1 \quad (7.1)$$

where $q = \exp(-g_s)$. These local operators, which coincide in the limit $q \rightarrow 0$; that is when g_s goes to ∞ , can be obtained from $\Gamma_{\pm}(z)$ by making the substitution

$$z^n \rightarrow \frac{z^n}{(1-q^n)^{p-1}}, \quad p \geq 2. \quad (7.2)$$

The $\Gamma_-^{(p)}$ s form then an infinite hierarchy of q-deformed vertex operators and obey commutation relations quite similar to those satisfied by the *level one* $\Gamma_-^{(1)}(z) = \Gamma_-(z)$. In particular we have,

$$\Gamma_+^{(1)}(1) \Gamma_-^{(p)}(q) = G_p(q) \Gamma_-^{(p)}(q) \Gamma_+^{(1)}(1), \quad (7.3)$$

where $G_p(q)$ is precisely the generalized p -dimension MacMahon function. We also have the following general relation,

$$\langle 0 | \Gamma_+^{(1)}(z_1) \Gamma_-^{(l+1)}(z_l) | 0 \rangle = \prod_{k_l=0} \cdots \prod_{k_1=0} \left[\prod_{k_0=0} \left(\frac{1}{\left(1 - q^{k_0+k_1+\dots+k_l} \frac{z_l}{z_1}\right)} \right) \right]. \quad (7.4)$$

This relation can be given an interpretation as l copies of $c = \infty$ free CFT₂ representations. Indeed, setting $\frac{z_l}{z_1} = q$, $q^{k_0+k_1+\dots+k_l} = Q_{\mathbf{k}} q^{k_0}$ with $Q_{\mathbf{k}} = q^{k_1+\dots+k_l}$ and

$$Z_2(Q_{\mathbf{k}}, q) = \prod_{k_0=0}^{\infty} \left(\frac{1}{(1 - Q_{\mathbf{k}} q^{k_0})} \right), \quad \mathbf{k} = (k_1, \dots, k_l), \quad (7.5)$$

we can put the right hand side of (7.4) like,

$$\prod_{(k_1, \dots, k_l)=\mathbf{0}}^{\infty} Z_2(Q_{\mathbf{k}}, q) \sim ([Z_2]^{\infty})^l. \quad (7.6)$$

This factorization suggests that, roughly, $G_{l+1}(q)$ could be interpreted as given by the product of l copies of infinite products of Z_2 . Since from 2d conformal free field theory view, each Z_2^{∞} copy should be described by a free field CFT₂ representation with $c = \infty$, the $G_{l+1}(q)$ partition function would then correspond to a central charge

$$c = k^l, \quad (7.7)$$

with $k \rightarrow \infty$.

(2) Using the above *level* p vertex operators, we have shown that the p -dimensional generalized MacMahon function is given by the following two-point correlation function

$$G_p(q) = \langle 0 | \Gamma_+^{(1)}(1) \Gamma_-^{(p)}(q) | 0 \rangle \quad (7.8)$$

(3) The *level* p vertex operators $\Gamma_-^{(p)}$ satisfy several remarkable properties, in particular they can be realized as condensates of vertex operators of *lower levels* as shown below,

$$\Gamma_-^{(p)}(z) = \Gamma_-^{(p-1)}(z) \Gamma_-^{(p)}(qz), \quad (7.9)$$

so that $G_p(q)$ can be defined as a particular $(p+1)$ -point correlation function as given below,

$$\mathcal{G}_{p+1} = \langle 0 | \mathcal{O}_0(x_0) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_p(x_p) | 0 \rangle, \quad (7.10)$$

where the $\mathcal{O}_j(x_j)$ are given by eqs(6.32-6.33). This correlation function can be expressed in different forms by using Wick theorem and the property (6.7).

(4) Based on the field theoretical derivation given in the present study, we learn that the function $G_p(q)$ with $p \geq 4$ cannot be the generating functional of the p -dimensional generalized Young diagrams.

Recall that for the case $p = 3$, solid partitions $\Pi^{(3)}$ extending Young diagrams have

generally three boundaries given by 2d partitions λ , μ and ν . The typical generating functional of all possible 3d partitions $\Psi^{(3)}$ with boundaries $\partial(\Psi^{(3)}) = (\lambda, \mu, \nu)$ is given by the correlation function $C_{\lambda\mu\nu}$

$$C_{\lambda\mu\nu} = \langle \nu^t | \mathcal{A}_+(\lambda) \mathcal{A}_-(\lambda^t) | \mu \rangle, \quad (7.11)$$

where $\mathcal{A}_+(\lambda) \mathcal{A}_-(\lambda^t)$ is the transfer matrix operator described previously. For the simplest case where $\partial(\Psi^{(3)}) = (\emptyset, \emptyset, \emptyset)$, the correlation function $C_{\emptyset\emptyset\emptyset}$ is precisely the generating functional of 3d partitions.

For higher values of p ; say $p = 4$, one has 4d generalized Young diagrams $\Psi^{(4)}$. This 4d partitions have generally *four* 3-dimensional *boundaries* captured by 3d partitions $\Lambda^{(3)}$, $\Sigma^{(3)}$, $\Upsilon^{(3)}$ and $\Pi^{(3)}$. The typical generating functional of all possible 4d partitions $\Pi^{(4)}$ with boundaries $\partial(\Psi^{(4)}) = (\Lambda^{(3)}, \Sigma^{(3)}, \Upsilon^{(3)}, \Pi^{(3)})$ is given by the correlation function $C_{\Lambda\Sigma\Upsilon\Psi}$. This functional extends (7.11) and can be defined as

$$\langle \langle \Lambda^{(3)} | \mathcal{A}'_-(\Sigma^{(3)}, \Upsilon^{(3)}) \mathcal{A}'_+(\Sigma^{(3)}, \Upsilon^{(3)}) | \Pi^{(3)} \rangle \rangle, \quad (7.12)$$

where $\mathcal{A}'_-(\Sigma^{(3)}, \Upsilon^{(3)}) \mathcal{A}'_+(\Sigma^{(3)}, \Upsilon^{(3)})$ is some generalized transfer matrix operator acting on 3d partition states $|\Pi^{(3)}\rangle$. It is this function that would generate the 4d generalized Young diagrams with boundaries Λ , Σ , Υ and Π .

Moreover, using the fact that 3d partitions $\Pi^{(3)}$ may themselves be sliced in terms of 2d partitions, one can usually bring the correlation function $C_{\Lambda\Sigma\Upsilon\Psi}$ to the form,

$$\langle \langle \varsigma, \tau, \nu | \mathcal{A}'_+[(\lambda, \mu, \nu); (\zeta, \eta, \theta)] \mathcal{A}'_-[(\lambda^t, \mu^t, \nu^t); (\zeta^t, \eta^t, \theta^t)] | \alpha, \beta, \gamma \rangle \rangle, \quad (7.13)$$

where $|\vartheta, \sigma, \varrho\rangle$ is a 3d partition boundary state expressed in terms of 2d partitions $|\vartheta\rangle \otimes |\sigma\rangle \otimes |\varrho\rangle$. In the particular case $\alpha = \beta = \gamma = \emptyset$ and $\varsigma = \tau = \nu = \emptyset$, the correlation function becomes

$$\langle \langle \emptyset, \emptyset, \emptyset | \mathcal{A}'_+[(\lambda, \mu, \nu); (\zeta, \eta, \theta)] \mathcal{A}'_-[(\lambda^t, \mu^t, \nu^t); (\zeta^t, \eta^t, \theta^t)] | \emptyset, \emptyset, \emptyset \rangle \rangle. \quad (7.14)$$

In the special case $\zeta = \eta = \theta = \lambda = \mu = \nu = \emptyset$, the above quantity simplifies as

$$\langle \langle \emptyset, \emptyset, \emptyset | \mathcal{A}'_+[\emptyset] \mathcal{A}'_-[\emptyset] | \emptyset, \emptyset, \emptyset \rangle \rangle. \quad (7.15)$$

From this general relation, we see that the MacMahon function $G_4(q) = \langle 0 | \Psi_+(1) \Omega_-(q) | 0 \rangle$ eq(4.10) appears as a very particular correlation function and then cannot be the generating functional of all possible 4d partitions.

Acknowledgement 1

This research work is supported by the program Protars III D12/25. HJ would like to thank ICTP for kind hospitality where part of this work has been done. The authors thank B. Szendroi for helpful suggestion.

8 Appendices

In this section, we give two appendices: an *Appendix A* where we describe the vertex operators $\Gamma_{\pm}^{(n)}(x)$ and their commutation relations algebra. An *Appendix B* which deals with the derivation of eq(5.17).

8.1 Appendix A: Vertex operators $\Gamma_{\pm}^{(n)}(x)$

We first study the *level n* vertex operators $\Gamma_{\pm}^{(n)}(x)$ and their main properties starting by $\Gamma_{\pm}^{(2)} = \Psi_{\pm}$. Then, we give their algebra.

8.1.1 Level 2 vertex operator

To begin notice that the operators $\Psi_{\pm}(1)$ eq(3.11), denoted also as $\Gamma_{\pm}^{(2)}(1)$, can be put in the form,

$$\begin{aligned}\Psi_{-}(1) &= \lim_{s \rightarrow \infty} \left[\left(\prod_{t=0}^s \Gamma_{-}(q^t) \right) q^{sL_0} \right], \\ \Psi_{+}(1) &= \lim_{s \rightarrow \infty} \left[q^{sL_0} \left(\prod_{t=0}^s \Gamma_{+}(q^{-t}) \right) \right].\end{aligned}\tag{8.1}$$

Using the expression of $\Gamma_{\pm}(z)$ eq(2.11), we can rewrite $\Psi_{\pm}(1)$ as follows:

$$\begin{aligned}\Psi_{-}(y) &= \left(\prod_{t=-\infty}^{-1} \Gamma_{-}(y) q^{L_0} \right) = \prod_{k=0}^{\infty} \Gamma_{-}(q^k y), \\ \Psi_{+}(x) &= \left(\prod_{t=0}^{\infty} q^{L_0} \Gamma_{+}(x) \right) = \prod_{t=0}^{\infty} \Gamma_{+}(q^{-k} x),\end{aligned}\tag{8.2}$$

or equivalently like

$$\begin{aligned}\Psi_{-}(y) &= \exp \left(\sum_{n \geq 1} \frac{i}{n} \frac{y^n}{(1 - q^n)} J_{-n} \right), \\ \Psi_{+}(x) &= \exp \left(- \sum_{n \geq 1} \frac{i}{n} \frac{x^{-n}}{(1 - q^n)} J_n \right).\end{aligned}\tag{8.3}$$

Let us compute the algebra of these vertex operators.

First, we have,

$$q^{L_0} \Psi_{\pm}(z) q^{-L_0} = \Psi_{\pm}(qz),\tag{8.4}$$

showing that q^{L_0} acts as a translation operator. We also have

$$\Psi_{\pm}(x) \Psi_{\pm}(y) = \Psi_{\pm}(y) \Psi_{\pm}(x).\tag{8.5}$$

To get the commutator between $\Psi_+(x)$ and $\Psi_-(y)$, we can do it in two ways which, by their comparison, allow us to get a new identity:

(i) Computation by using products of Γ_{\pm} . We have,

$$\begin{aligned}\Psi_+(x) \Psi_-(y) &= \prod_{l \geq 0} \Gamma_+(q^l x) \prod_{k \geq 0} \Gamma_-(q^k y) \\ &= \prod_{s=0}^{\infty} \left(1 - q^s \frac{y}{x}\right)^{-(s+1)} \Psi_-(y) \Psi_+(x),\end{aligned}\quad (8.6)$$

in particular

$$\Psi_+(1) \Psi_-(q) = \left(\prod_{t=1}^{\infty} (1 - q^t)^{-t} \right) \Psi_-(q) \Psi_+(1). \quad (8.7)$$

Notice

$$\begin{aligned}\Gamma_+(1) \Psi_-(y) &= \Gamma_+(1) \prod_{k \geq 0} \Gamma_-(q^k y) \\ &= \prod_{k \geq 0} (1 - q^k y)^{-1} \Psi_-(y) \Gamma_+(1)\end{aligned}\quad (8.8)$$

and also

$$\begin{aligned}\Gamma_+(x) \Psi_-(y) &= \Gamma_+(x) \prod_{k \geq 0} \Gamma_-(q^k y) \\ &= \prod_{k \geq 0} \left(1 - q^k \frac{y}{x}\right)^{-1} \Psi_-(y) \Gamma_+(x).\end{aligned}\quad (8.9)$$

We also have

$$\begin{aligned}\Psi_+(x) \Gamma_-(1) &= \prod_{k \geq 0} \Gamma_+(q^{-k} x) \Gamma_-(1) \\ &= \prod_{k \geq 0} (1 - q^k x^{-1})^{-1} \Gamma_-(1) \Psi_+(x)\end{aligned}\quad (8.10)$$

(ii) Computation using directly eqs(8.3). We get,

$$\Psi_+(x) \Psi_-(y) = \exp \left(\sum_{n \geq 1} \frac{1}{n} \frac{y^n x^{-n}}{(1 - q^n)^2} \right) \Psi_-(y) \Psi_+(x). \quad (8.11)$$

By comparing the two expressions (8.6) and (8.11), we get the following identity

$$\exp \left(\sum_{n \geq 1} \frac{1}{n} \frac{y^n x^{-n}}{(1 - q^n)^2} \right) = \prod_{s=0}^{\infty} \left(1 - q^s \frac{y}{x}\right)^{-(s+1)}. \quad (8.12)$$

or equivalently like,

$$\sum_{n \geq 1} \frac{1}{n} \frac{y^n x^{-n}}{(1 - q^n)^2} = - \sum_{s=0}^{\infty} \left[(s+1) \ln \left(1 - q^s \frac{y}{x}\right) \right]$$

8.1.2 Generic q-deformed operators

Here we give the expressions of the generic q-deformed operators and some useful properties of their algebra.

The starting point is the vertex operators

$$\Gamma_{-}(z) = \exp \left(i \sum_{n>0} \frac{1}{n} z^n J_{-n} \right), \quad \Gamma_{+}(z) = \exp \left(-i \sum_{n>0} \frac{1}{n} z^{-n} J_n \right). \quad (8.13)$$

and the aim is:

(1) compute for $n \geq 1$, the following hierarchy of composite vertex operators

$$\Gamma_{-}^{(n+1)}(z) = \left(\prod_{t_n=1}^{\infty} \cdots \left[\prod_{t_2=1}^{\infty} \left(\prod_{t_1=1}^{\infty} \Gamma_{-}(z) q^{L_0} \right) q^{L_0} \right] \cdots q^{L_0} \right). \quad (8.14)$$

For $n = 0$, we have just $\Gamma_{-}^{(1)}(z) = \Gamma_{-}(z)$. Similar quantities can be written down for $\Gamma_{+}^{(n+1)}(z)$; we shall not report them here.

(2) derive the identity (6.30).

For these purposes, we proceed by using inductive method:

q- deformed vertex operators $\Gamma_{-}^{(2)}(z)$ and $\Gamma_{-}^{(3)}(z)$:

(a) **Case $\Gamma_{-}^{(2)}(z)$:**

In this case, we have

$$\Gamma_{-}^{(2)}(z) = \prod_{t=1}^{\infty} (\Gamma_{-}(z) q^{L_0}). \quad (8.15)$$

Notice that

$$\Gamma_{-}^{(2)}(z) = \lim_{s \rightarrow \infty} \prod_{t=1}^s (\Gamma_{-}(z) q^{L_0}). \quad (8.16)$$

By using the fact that q^{L_0} acts as translation operator on $\Gamma_{-}(z)$, we get

$$\Gamma_{-}^{(2)}(z) = \lim_{s \rightarrow \infty} \left(\left[\prod_{k=0}^s (\Gamma_{-}(q^k z)) \right] q^{sL_0} \right). \quad (8.17)$$

For simplicity, we consider the action of $\Gamma_{-}^{(2)}(z)$, on the vacuum, that reads as

$$\Gamma_{-}^{(2)}(z) |0\rangle = \left(\prod_{k=0}^{\infty} \Gamma_{-}(q^k z) \right) |0\rangle. \quad (8.18)$$

Substituting $\Gamma_{-}(q^k z)$ by its expression (8.13), we find

$$\begin{aligned} \Gamma_{-}^{(2)}(z) |0\rangle &= \exp \left(i \sum_{n>0} \sum_{k=0}^{\infty} \frac{q^{kn}}{n} z^n J_{-n} \right) |0\rangle \\ &= \exp \left(i \sum_{n>0} \frac{1}{n} \frac{z^n}{(1-q^n)} J_{-n} \right) |0\rangle. \end{aligned} \quad (8.19)$$

Notice that $\Gamma_-^{(2)}(z)|0\rangle$ can be decomposed as follows,

$$\Gamma_-^{(2)}(z)|0\rangle = \exp\left(i\sum_{n>0}\frac{1}{n}z^n J_{-n}\right) \exp\left(i\sum_{n>0}q^n\sum_{k=0}^{\infty}\frac{q^{kn}}{n}z^n J_{-n}\right)|0\rangle \quad (8.20)$$

or equivalently like

$$\Gamma_-^{(2)}(z)|0\rangle = \exp\left(i\sum_{n>0}\frac{1}{n}z^n J_{-n}\right) \exp\left(i\sum_{n>0}\frac{1}{n}\frac{(qz)^n}{(1-q^n)}J_{-n}\right)|0\rangle \quad (8.21)$$

showing that we have:

$$\Gamma_-^{(2)}(z)|0\rangle = \Gamma_-^{(1)}(z)\Gamma_-^{(2)}(qz)|0\rangle, \quad z \in C. \quad (8.22)$$

(b) Case $\Gamma_-^{(3)}(z)$:

Here, we have

$$\Gamma_-^{(3)}(z)|0\rangle = \prod_{t_2=1}^{\infty} \left[\Gamma_-^{(2)}(z)q^{L_0} \right] |0\rangle = \prod_{k=0}^{\infty} \left(\Gamma_-^{(2)}(q^k z) \right) |0\rangle. \quad (8.23)$$

Substituting $\Gamma_-^{(2)}(q^k z)$ by its expression (8.2), we find

$$\begin{aligned} \Gamma_-^{(3)}(z)|0\rangle &= \exp\left(i\sum_{n>0}\frac{1}{n}\sum_{k=0}^{\infty}q^{kn}\frac{z^n}{(1-q^n)}J_{-n}\right)|0\rangle \\ &= \exp\left(i\sum_{n>0}\frac{1}{n}\frac{z^n}{(1-q^n)^2}J_{-n}\right)|0\rangle \end{aligned} \quad (8.24)$$

Finally, if we rewrite the above relation as follows

$$\Gamma_-^{(3)}(z)|0\rangle = \exp\left(\sum_{n>0}\frac{iz^n J_{-n}}{n(1-q^n)}\right) \exp\left(\sum_{n>0}\frac{i(qz)^n J_{-n}}{n(1-q^n)^2}\right)|0\rangle \quad (8.25)$$

we obtain the relation

$$\Gamma_-^{(3)}(z)|0\rangle = \Gamma_-^{(2)}(z)\Gamma_-^{(3)}(qz)|0\rangle \quad (8.26)$$

as claimed in section 6 eq(6.18).

Higher levels

From the above analysis, it is not difficult to check that the explicit expression of the vertex operators $\Gamma_-^{(p)}(z)$ acting on the vacuum is given by,

$$\Gamma_-^{(p)}(z)|0\rangle = \exp\left(\sum_{n=1}^{\infty}\frac{iz^n J_{-n}}{n(1-q^n)^{p-1}}\right)|0\rangle, \quad p \geq 1. \quad (8.27)$$

Moreover using the identity,

$$\frac{z^n}{(1-q^n)^{p-1}} = \sum_{k=1}^{p-1} \frac{(qz)^n}{(1-q^n)^{k-1}}, \quad (8.28)$$

we can decompose the above relation as follows

$$\Gamma_-^{(p)}(z) |0\rangle = \Gamma_-^{(1)}(z) \prod_{k=2}^{p-1} \Gamma_-^{(k)}(qz) |0\rangle. \quad (8.29)$$

The next step is to use the relation

$$\frac{z^n}{(1-q^n)^{p-1}} = \frac{z^n}{(1-q^n)^{p-2}} + \frac{(qz)^n}{(1-q^n)^{p-1}}, \quad (8.30)$$

that imply the equality

$$\Gamma_-^{(p)}(z) |0\rangle = \Gamma_-^{(p-1)}(z) \Gamma_-^{(p)}(qz) |0\rangle. \quad (8.31)$$

Doing the same for the first term for the right hand side

$$\frac{z^n}{(1-q^n)^{p-2}} = \frac{z^n}{(1-q^n)^{p-3}} + \frac{(qz)^n}{(1-q^n)^{p-2}}, \quad (8.32)$$

the eq(8.30) can be brought to the form

$$\frac{z^n}{(1-q^n)^{p-1}} = \frac{z^n}{(1-q^n)^{p-3}} + \frac{(qz)^n}{(1-q^n)^{p-2}} + \frac{(qz)^n}{(1-q^n)^{p-1}}, \quad (8.33)$$

leading then to

$$\Gamma_-^{(p)}(z) |0\rangle = \Gamma_-^{(p-2)}(z) \Gamma_-^{(p-1)}(qz) \Gamma_-^{(p)}(qz) |0\rangle. \quad (8.34)$$

We can repeat this operation successively to end with the two following:

(i) the expression of level p vertex operator as we have used it in section 6 eq(6.32),

$$Z_{pd} = \langle 0 | \mathcal{T} | 0 \rangle, \quad (8.35)$$

with

$$\mathcal{T} = \Gamma_+^{(1)}(z) \prod_{i=1}^{p_j} \Gamma_-^{(j+1)}(q^j z), \quad (8.36)$$

where p_j are given by

$$p_j = \frac{(p-1)!}{j!(p-j-1)!}, \quad j = 0, \dots, p-1. \quad (8.37)$$

(ii) Using the decomposition for $\Gamma_-^{(p)}(z)$,

$$\Gamma_-^{(p)}(z) = \mathcal{O}_0(x_0) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_{p-1}(x_{p-1}) \quad (8.38)$$

we get

$$\mathcal{O}_{j+1}(x_{j+1}) = \prod_{i=1}^{p_j} \Gamma_-^{(j+1)}(q^j z), \quad j = 0, \dots, p-1. \quad (8.39)$$

8.2 Appendix B: Combinatorial eq(5.17)

Here we want to derive the identity (5.17) namely,

$$\sum_{k=1}^s C_{k+p-3}^{p-2} = C_{s+p-2}^{p-1}, \quad p \geq 2. \quad (8.40)$$

This is a standard combinatorial identity; its proof follows from basic property [37],

$$C_{n+1}^k = C_n^{k-1} + C_n^k. \quad (8.41)$$

Applying this identity to C_n^k and putting it back into the above relation, we get,

$$C_{n+1}^k = C_n^{k-1} + C_{n-1}^{k-1} + C_{n-1}^k. \quad (8.42)$$

By induction, it results,

$$C_{n+1}^k = \sum_{j=k-1}^n C_j^{k-1}. \quad (8.43)$$

Setting $k = p - 1$ and $n = s + p - 3$, we recover the identity (8.40).

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